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Miscellaneous Sharp Inequalities and Korovkin-Type Convergence Theorems Involving Sequences of Probability Measures*

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We generalize a theorem due to P. P. Korovkin (see [1]) to sequences of arbitrary probability measures on $[0, \pi]$. Korovkin's result is concerned with the convergence of certain ratios of the Fourier coefficients of a sequence of density functions. Earlier, E. L. Stark (see [2]) gave a different generalization of this Korovkin theorem.

Analogous characterizations are given for the same type of ratios of the hyperbolic coefficients (respectively, the Laplace transforms) of a sequence of probability measures on $\mathbb{R}$ (respectively, on $\mathbb{R}^+$). In the course of the proofs we establish various inequalities, on subsets of $\mathbb{R}$, leading to several sharp estimates. A number of related applications are given.

The following is the basis for the next convergence results.

LEMMA 1. *For $k, l \geqslant 2$, $k, l \in \mathbb{N}$ there exists a positive constant $C(k, l) \geqslant [k^2(k^2-1)]/[l^2(l^2-1)]$ such that*

$$[k^2(1-\cos t)-(1-\cos kt)] \leqslant C(k, l)[l^2(1-\cos t)-(1-\cos lt)], \text{ all } t \in [0, \pi]. \tag{1.1}$$

Proof. Since $|\sin nt| \leqslant n|\sin t|$, $n \in \mathbb{N}$, we have that for $t \in (0, \pi]$

$$n^2(1-\cos t)-(1-\cos nt) = \left(n^2 - \frac{\sin^2(nt/2)}{\sin^2(t/2)}\right)(1-\cos t) \geqslant 0.$$

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The function

$$\varphi(t)=\frac{k^2(1-\cos t)-(1-\cos kt)}{l^2(1-\cos t)-(1-\cos lt)}$$

on $[0, \pi]$ satisfies

$$\lim_{t\to 0}\varphi(t)=\frac{k^2(k^2-1)}{l^2(l^2-1)}>0,$$

so that $\varphi(t)$ is strictly positive and continuous, therefore bounded. ∎

Remark. We conjecture that

$$C(k, l)=k^2(k^2-1)\,l^{-2}(l^2-1)^{-1} \quad \text{if } 2\leqslant l\leqslant k.$$

It is correct for $k\leqslant 5$ and the corresponding inequality is sharp.

DEFINITION 2. Let μ be a probability measure on $[0,\pi]$. Its Fourier–Stieltjes coefficients are defined as

$$p_k=\int_0^{\pi}\cos kt\mu(dt) \qquad (k=0, 1, 2,...). \tag{2.1}$$

If $p_1=1$, then $\mu=\delta_0$.

LEMMA 3. *Let μ be a probability measure on $[0,\pi]$ with Fourier–Stieltjes coefficients p_k, $k\in\mathbb{Z}^+$ and $p_1\neq 1$.*
Then

$$\left[k^2-\left(\frac{1-p_k}{1-p_1}\right)\right]\leqslant C(k, l)\left[l^2-\left(\frac{1-p_l}{1-p_1}\right)\right], \tag{3.1}$$

where k, $l\geqslant 2$, k, $l\in\mathbb{N}$.

Proof. Integrate (1.1) relative to μ and divide both sides by $(1-p_1)$. ∎

THEOREM 4. *Let $l\in\mathbb{N}$, $l\geqslant 2$. If $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of probability measures on $[0,\pi]$ with Fourier–Stieltjes coefficients p_{kn} such that $p_{1n}\neq 1$ and $\lim_{n\to\infty}((1-p_{ln})/(1-p_{1n}))=l^2$, then $\lim_{n\to\infty}((1-p_{kn})/(1-p_{1n}))=k^2$ for all $k\geqslant 2$, $k\in\mathbb{N}$.*

Proof. Use (3.1). ∎

Remark 5. In the sequel k, $l\in\mathbb{N}$ and k, $l\geqslant 2$. Let $g(s)$ be the characteristic function (Fourier transform) of a probability measure μ on $[0, \pi]$. We have that

$$\operatorname{Re}(1-g(s)) = \int_0^{\pi} (1-\cos st)\,\mu(dt), \qquad s \in \mathbb{R}.$$

Then by applying (1.1) we get

$$[k^2(1-\operatorname{Re} g(1)) - (1-\operatorname{Re} g(k))] \leqslant C(k,l)[l^2(1-\operatorname{Re} g(1)) - (1-\operatorname{Re} g(l))].$$

As an illustration, let $g(s) = |f(s)|^2$, where f is a characteristic function. Then also g is a characteristic function. It follows that

$$[k^2(1-|f(1)|^2) - (1-|f(k)|^2)] \leqslant C(k,l)[l^2(1-|f(1)|^2) - (1-|f(l)|^2)].$$

Consequently, if a sequence $\{f_n\}_{n \in \mathbb{N}}$ of characteristic functions with $|f_n(1)| < 1$ satisfies

$$\lim_{n \to \infty} \left(\frac{1-|f_n(k)|^2}{1-|f_n(1)|^2} \right) = k^2 \quad \text{for one } k \in \mathbb{N},\ k \geqslant 2$$

then for all such k.

Now we proceed to a similar type of result.

LEMMA 6. *Let* $k \geqslant l > 1$, $B(k,l) = l^2(l^2-1)\,k^{-2}(k^2-1)^{-1}$. *Then*

$$[(\operatorname{Cos} hlt - 1) - l^2(\operatorname{Cos} ht - 1)] \leqslant B(k,l)[(\operatorname{Cos} hkt - 1) - k^2(\operatorname{Cos} ht - 1)] \tag{6.1}$$

for all $t \in \mathbb{R}$.

Here the constant $B(k, l)$ *cannot be improved, i.e.,* (6.1) *is sharp.*

Proof. Easy, namely, by writing (6.1) as

$$\sum_{j=2}^{\infty} \frac{t^{2j}}{(2j)!}(l^{2j} - l^2) \leqslant B(k,l) \sum_{j=2}^{\infty} \frac{t^{2j}}{(2j)!}(k^{2j} - k^2).$$

The last assertion follows by

$$A_\gamma(t) = [(\operatorname{Cos} h\gamma t - 1) - \gamma^2(\operatorname{Cos} ht - 1)] \geqslant 0, \qquad \gamma > 1$$

and

$$\lim_{t \to 0} (A_l/A_k) = B(k,l). \quad \blacksquare$$

DEFINITION 7. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$ such that the following integrals exist:

$$\tilde{p}_{k,n}=\int_{\mathbb{R}} \operatorname{Cos} hkt\mu_n(dt) \qquad (k\in\mathbb{R}). \tag{7.1}$$

We shall call the numbers $\tilde{p}_{k,n}$ hyperbolic coefficients of the measure μ_n.

Now we have:

THEOREM 8. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$ with $\tilde{p}_{k,n}<\infty$ and $\tilde{p}_{1,n}\neq 1$. Let $k>1$ and suppose that*

$$\lim_{n\to\infty}\frac{\tilde{p}_{k,n}-1}{\tilde{p}_{1,n}-1}=k^2,$$

then

$$\lim_{n\to\infty}\frac{\tilde{p}_{l,n}-1}{\tilde{p}_{1,n}-1}=l^2 \qquad \text{for all } 1<l\leqslant k.$$

Proof. Immediate from an integration of (6.1) with respect to μ_n. ∎

The next result is a characterization of the convexity of functions and leads to some applications as it is Proposition 11.

LEMMA 9. *If $f\colon (0,\infty)\to\mathbb{R}$ then $h(x)=f(x)/x$ is convex iff $\varphi(k)=(k^2f(x)-f(kx))/(k^2-k)$ is non-increasing in $k>1$, for each $x>0$. Equivalently if $\psi(k)=(kf(x)-f(kx))/(k^2-k)$ is non-increasing in $k>1$, for each $x>0$.*

Proof. Let us first assume that h is convex. Consider $1<\lambda<k$. We have

$$\lambda x=\left(\frac{k-\lambda}{k-1}\right)x+\left(\frac{\lambda-1}{k-1}\right)kx, \qquad \text{with } \frac{k-\lambda}{k-1}+\frac{\lambda-1}{k-1}=1,$$

where both $(k-\lambda)/(k-1)$, $(\lambda-1)/(k-1)>0$. Since h is convex, one has

$$h(\lambda x)\leqslant\left(\frac{k-\lambda}{k-1}\right)h(x)+\left(\frac{\lambda-1}{k-1}\right)h(kx). \tag{9.1}$$

Substituting $h(x)=f(x)/x$ one obtains precisely $\varphi(\lambda)\geqslant\varphi(k)$. Next, assume that φ is non-increasing for each $x>0$. This is equivalent to (9.1). Now suppose $0<x_1<x_2$ and let $\alpha=x_2/x_1>1$. Applying (9.1) with $\lambda=(1+\alpha)/2$, $k=\alpha$, and $x=x_1$ one has

$$h\left(\frac{x_1+x_2}{2}\right)=h\left(\left(\frac{1+\alpha}{2}\right)x_1\right)$$
$$\leqslant\left(\frac{\alpha-((1+\alpha)/2)}{\alpha-1}\right)h(x_1)+\left(\frac{((1+\alpha)/2)-1}{\alpha-1}\right)h(\alpha x_1)$$
$$=\frac{1}{2}h(x_1)+\frac{1}{2}h(x_2).$$

Therefore h is convex.

Next we prove that $\varphi(k)$ being non-increasing in $k>1$ for each $x>0$ is equivalent to $\psi(k)$ being non-increasing. In other words, we want to show that for $1<\lambda<k$

$$\frac{[\lambda^2 f(x)-f(\lambda x)]}{(\lambda^2-\lambda)}-\frac{[k^2 f(x)-f(kx)]}{(k^2-k)}\geqslant 0$$

is equivalent to

$$\frac{[\lambda f(x)-f(\lambda x)]}{(\lambda^2-\lambda)}-\frac{[kf(x)-f(kx)]}{(k^2-k)}\geqslant 0.$$

In fact, the difference between the two left-hand sides equals

$$\frac{(\lambda^2-\lambda)f(x)}{(\lambda^2-\lambda)}-\frac{(k^2-k)f(x)}{(k^2-k)}=0. \quad ∎$$

Remark 10. Let f: $(0,\infty)\to\mathbb{R}$ such that $f(x)/x$ is convex. Then by Lemma 9 one has for $1<\lambda<k$, all $x>0$, the two equivalent inequalities

$$(k^2 f(x)-f(kx))\leqslant\left(\frac{k^2-k}{\lambda^2-\lambda}\right)(\lambda^2 f(x)-f(\lambda x)) \tag{10.1}$$

and

$$(kf(x)-f(kx))\leqslant\left(\frac{k^2-k}{\lambda^2-\lambda}\right)(\lambda f(x)-f(\lambda x)). \tag{10.2}$$

Quantities such as $[\lambda f(x)-f(\lambda x)]$ can be regarded as a measure of linearity for f.

PROPOSITION 11. *For $k>\lambda>1$ and $x>0$ obtain the following two equivalent sharp inequalities*:

$$k^2(1-e^{-x})-(1-e^{-kx})<\left(\frac{k^2-k}{\lambda^2-\lambda}\right)(\lambda^2(1-e^{-x})-(1-e^{-\lambda x})) \tag{11.1}$$

and

$$k(1-e^{-x})-(1-e^{-kx})<\left(\frac{k^2-k}{\lambda^2-\lambda}\right)(\lambda(1-e^{-x})-(1-e^{-\lambda x})). \quad (11.2)$$

Proof. Note that the function $f(x)=1-e^{-x}$, $x\in(0,+\infty)$, satisfies

$$\frac{f(x)}{x}=\int_0^1 e^{-\theta x}\,d\theta$$

showing that $f(x)/x$ is convex. By Lemma 9 we have (10.1) and (10.2). These are precisely (11.1) and (11.2). The sharpness follows from

$$\lim_{x\to 0}\frac{k^2(1-e^{-x})-(1-e^{-kx})}{\lambda^2(1-e^{-x})-(1-e^{-\lambda x})}=\lim_{x\to 0}\frac{k(1-e^{-x})-(1-e^{-kx})}{\lambda(1-e^{-x})-(1-e^{-\lambda x})}$$
$$=\left(\frac{k^2-k}{\lambda^2-\lambda}\right). \quad ∎$$

Some applications of the last proposition are Theorem 14 and Proposition 16 following.

DEFINITION 12. Let μ be a probability measure on $\mathbb{R}^+$. For $\lambda\geqslant 0$ its Laplace transform is defined as

$$\varphi(\lambda)=\int_0^\infty e^{-\lambda t}\mu(dt). \quad (12.1)$$

LEMMA 13. *Let $k\geqslant\lambda>1$ and let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}^+$ with existing Laplace transforms φ_n such that $\varphi_n(1)\neq 1$ all $n\in\mathbb{N}$. Then one has the equivalent inequalities*

$$\left[k^2-\left(\frac{1-\varphi_n(k)}{1-\varphi_n(1)}\right)\right]\leqslant\left(\frac{k^2-k}{\lambda^2-\lambda}\right)\left[\lambda^2-\left(\frac{1-\varphi_n(\lambda)}{1-\varphi_n(1)}\right)\right] \quad (13.1)$$

and

$$\left[k-\left(\frac{1-\varphi_n(k)}{1-\varphi_n(1)}\right)\right]\leqslant\left(\frac{k^2-k}{\lambda^2-\lambda}\right)\left[\lambda-\left(\frac{1-\varphi_n(\lambda)}{1-\varphi_n(1)}\right)\right]. \quad (13.2)$$

Proof. Integrate (11.1) and (11.2) relative to μ_n and divide by $(1-\varphi_n(1))>0$. ∎

THEOREM 14. *Let $\lambda>1$. Then*

$$\lim_{n\to\infty}\left(\frac{1-\varphi_n(\lambda)}{1-\varphi_n(1)}\right)=\lambda^2 \qquad \text{(respectively } \lambda\text{)}$$

implies that

$$\lim_{n \to \infty} \left(\frac{1-\varphi_n(k)}{1-\varphi_n(1)} \right) = k^2 \quad \text{(respectively } k\text{) for all } k \geqslant \lambda.$$

Proof. Apply (13.1) (respectively (13.2)). ∎

DEFINITION 15. The Weierstrass operator is the positive linear operator defined by

$$(W_n f)(t) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-n(t-x)^2}\, dx, \quad \text{all } n \in \mathbb{N},$$

where $f \in C_B(\mathbb{R})$.

One has $W_n f \xrightarrow{u} f$ as $n \to \infty$.

PROPOSITION 16. *Let $f \in C_B(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |f(x)|^n\, dx = C_n \in (0, \infty)$; all $n \in \mathbb{N}$. Consider $k \geqslant \lambda$ with $k, \lambda \in \mathbb{N}$.*

If

$$\lim_{n \to \infty} \left[\frac{1 - \sqrt{\frac{\pi}{\lambda}}\, W_\lambda \left(\frac{|f|^n}{C_n}, t \right)}{1 - \sqrt{\pi}\, W_1 \left(\frac{|f|^n}{C_n}, t \right)} \right] = \lambda^2 \qquad (\textit{respectively } \lambda)$$

then

$$\lim_{n \to \infty} \left[\frac{1 - \sqrt{\frac{\pi}{k}}\, W_k \left(\frac{|f|^n}{C_n}, t \right)}{1 - \sqrt{\pi}\, W_1 \left(\frac{|f|^n}{C_n}, t \right)} \right] = k^2 \qquad (\textit{respectively } k).$$

Proof. Apply (11.1) (respectively (11.2)) with x replaced by $(t-x)^2$, where t is fixed. Afterwards multiply by $|f|^n$ and integrate over $\mathbb{R}$. ∎

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